CALCULATION OF THREE-DIMENSIONAL ROTATIONAL SUPERSONIC FLOW OF GAS IN THE VICINITY OF A CURVE AT WHICH OCCURS A BREAK IN STREAMLINE

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PMM Vol.27, No.6, 1963, pp.1083-1089

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(Received July 1, 1963)

We consider the problem of finding the three-dimensional supersonic flow of an ideal gas in the vicinity of a curve L, at points of which there occurs a break in the streamlines. The solution of the three-dimensional equations of gas motion is sought in a special coordinate system in the form of a series in one of the variables with coefficients depending upon the other two variables. For the determination of the coefficients we obtain a recurrent system of ordinary differential equations.

The equations for the zero-order terms in the series have two solutions, which correspond to flows with expansion or compression in the vicinity of the curve L.

In the case when L is a circumference or an arc of a circumference, the solution of the recurrent system of equations is found in the form of quadratures.

1. In a cylindrical coordinate system x, y and φ (see Figure) the equations of three-dimensional rotational flow of a gas have the form

$$\left(1-\frac{u^2}{c^2}\right)\frac{\partial u}{\partial x}+\left(1-\frac{v^2}{c^2}\right)\frac{\partial v}{\partial y}+\left(1-\frac{w^2}{c^2}\right)\frac{1}{y}\frac{\partial w}{\partial \varphi}-$$

$$-\frac{uv}{c^2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)-\frac{uv}{c^2}\left(\frac{\partial w}{\partial x}+\frac{1}{y}\frac{\partial u}{\partial \varphi}\right)-\frac{vw}{c^2}\left(\frac{\partial w}{\partial y}+\frac{1}{y}\frac{\partial v}{\partial \varphi}\right)+\frac{v}{y}=0$$

$$v\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+w\left(\frac{1}{y}\frac{\partial u}{\partial \varphi}-\frac{\partial w}{\partial x}\right)=\frac{1-V^2}{2\varkappa}\frac{\partial S}{\partial x}$$

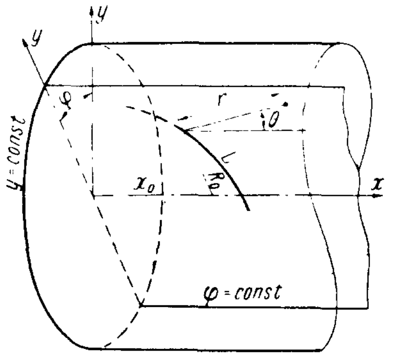
$$v\left(\frac{\partial w}{\partial y}+\frac{w}{y}-\frac{1}{y}\frac{\partial v}{\partial \varphi}\right)-u\left(\frac{1}{y}\frac{\partial u}{\partial \varphi}-\frac{\partial w}{\partial x}\right)=\frac{1-V^2}{2\varkappa}\frac{1}{y}\frac{\partial S}{\partial \varphi}$$

$$u\frac{\partial S}{\partial x}+v\frac{\partial S}{\partial y}+\frac{w}{y}\frac{\partial S}{\partial \varphi}=0,\qquad \frac{c^2={}^1/_2\,(\varkappa-1)\,(1-V^2)}{S=\log\,(p\,/\,\rho^{\varkappa})+\text{const}} \tag{1.1}$$

Here the velocity components u, v and w in the directions of x, y and φ and the speed of sound c are referred to the maximum value of velocity in the gas; p is the pressure, ρ the density and κ the ratio of specific heats for the gas.

We transform (1.1) to a new system of coordinates according the the equations

$$x=r\cos\theta+x_0(\varphi),\qquad y=r\sin\theta+R_0(\varphi),\quad \Phi=\varphi \tag{1.2}$$



Here $x=x_0(\varphi)$, $y=R_0(\varphi)$ are the equations of a certain curve L, and r and θ are polar coordinates in the plane φ = const with origin at points of the curve L (see Figure).

Velocity components in the old and new coordinate systems are connected by the relations

$$u=v_r\cos\theta-v_\theta\sin\theta,\qquad v=v_r\sin\theta+v_\theta\cos\theta,\qquad w=w \tag{1.3}$$

where v_r and v_θ are the velocity components in the r and θ directions respectively.

By means of (1.2) we obtain

$$\frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta},\qquad \frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial\varphi}=\frac{\partial}{\partial\varphi}-B'\frac{\partial}{\partial r}+\frac{B}{r}\frac{\partial}{\partial\theta},\qquad B=\dot{x_0}\sin\theta-\dot{R_0}\cos\theta \tag{1.4}$$

Here dots denote derivatives with respect to φ and primes with respect to θ.

In the new coordinate system equations (1.1) take the form ($q^2 = {v_r}^2+{v_\theta}^2$)

$$w\left(\frac{\partial v_r}{\partial\theta}-B'\frac{\partial v_r}{\partial r}\right)-(R_0+r\sin\theta)\left(w\frac{\partial w}{\partial r}+v_\theta\frac{\partial v_\theta}{\partial r}+\frac{1-V^2}{2\varkappa}\frac{\partial S}{\partial r}\right)+ \tag{1.5}$$

$$+\left(\frac{\partial v_r}{\partial\theta}-v_\theta\right)\left[\frac{Bw}{r}+v_\theta\left(\frac{R_0}{r}+\sin\theta\right)\right]-w^2\sin\theta=0$$

$$\left(\frac{v_\theta R_0+Bw}{r}+v_\theta\sin\theta\right)\frac{\partial S}{\partial\theta}+[v_r(R_0+r\sin\theta)-B'w]\frac{\partial S}{\partial r}+w\frac{\partial S}{\partial\varphi}=0$$

$$(R_0+r\sin\theta)\,v_r\frac{\partial w}{\partial r}+\frac{B'}{2}\left(\frac{\partial q^2}{\partial r}+\frac{1-V^2}{\varkappa}\frac{\partial S}{\partial r}\right)+\left(\frac{R_0}{r}+\sin\theta\right)v_\theta\frac{\partial w}{\partial\theta}-$$
$$-\frac{B}{2r}\left(\frac{\partial q^2}{\partial\theta}+\frac{1-V^2}{\varkappa}\frac{\partial S}{\partial\theta}\right)-\frac{1}{2}\left(\frac{\partial q^2}{\partial\varphi}+\frac{1-V^2}{\varkappa}\frac{\partial S}{\partial\varphi}\right)+w\,(v_r\sin\theta+v_\theta\cos\theta)=0$$

$$\left(\frac{R_0}{r}+\sin\theta\right)\left[c^2\left(v_r+\frac{\partial v_\theta}{\partial\theta}\right)-\frac{v_\theta}{2}\frac{\partial q^2}{\partial\theta}-wv_\theta\frac{\partial w}{\partial\theta}\right]+$$
$$+\frac{B}{r}\left[(c^2-w^2)\frac{\partial w}{\partial\theta}-\frac{w}{2}\frac{\partial q^2}{\partial\theta}\right]+(R_0+r\sin\theta)\left(c^2\frac{\partial v_r}{\partial r}-\frac{v_r}{2}\frac{\partial q^2}{\partial r}-wv_r\frac{\partial w}{\partial r}\right)-$$
$$-B'\left[(c^2-w^2)\frac{\partial w}{\partial r}-\frac{w}{2}\frac{\partial q^2}{\partial r}\right]+(c^2-w^2)\frac{\partial w}{\partial\varphi}-\frac{w}{2}\frac{\partial q^2}{\partial\varphi}+c^2(v_r\sin\theta+v_\theta\cos\theta)=0$$

2. We seek the solution of system (1.5) in the form of series

$$v_r=\sum_{n=0}^{\infty}u_n r^n,\quad v_\theta=\sum_{n=0}^{\infty}v_n r^n,\quad w=\sum_{n=0}^{\infty}w_n r^n,\quad S=\sum_{n=0}^{\infty}S_n r^n \tag{2.1}$$
$$(u_n=u_n(\theta,\varphi),\ \ v_n=v_n(\theta,\varphi),\ \ w_n=w_n(\theta,\varphi),\ S_n=S(\theta,\varphi))$$

Substituting series (2.1) into system (1.5) and equating coefficients, we obtain

$$\sum_{i+j=n}(R_0v_i+Bw_i)\frac{\partial S_j}{\partial\theta}+\sum_{i+j=n-1}\left[v_i\sin\theta\frac{\partial S_j}{\partial\theta}+(j+1)(u_iR_0-B'w_i)\,S_{j+1}+w_i\frac{\partial S_i}{\partial\varphi}\right]+$$
$$+\sum_{i+j=n-2}(j+1)\,u_iS_{j+1}\sin\theta=0 \tag{2.2}$$

$$\sum_{i+j=n}\left(\frac{\partial u_i}{\partial\theta}-v_i\right)(Bw_j+R_0v_j)+\sum_{i+j+k=n-1}\left\{w_i\left[\frac{\partial u_j}{\partial\varphi}-B'(1+j)\,u_{j+1}-\right.\right.$$
$$\left.\left.-w_j\sin\theta\right]-R_0a_{ijk}+\left(\frac{\partial u_i}{\partial\theta}-v_i\right)v_j\sin\theta\right\}-\sum_{i+j+k=n-2}a_{ijk}\sin\theta=0 \tag{2.3}$$

$$\sum_{i+j+k=n}\left[R_0v_i\frac{\partial w_j}{\partial\theta}-\frac{B}{2}\frac{\partial q_{ij}}{\partial\theta}-\frac{B(1-V_{ij})}{2\varkappa}\frac{\partial S_k}{\partial\theta}\right]+\sum_{i+j+k=n-1}[(j+1)\,R_0u_iw_{j+1}+$$
$$+v_i\frac{\partial w_j}{\partial\theta}\sin\theta-\frac{1}{2}\frac{\partial q_{ij}}{\partial\varphi}-\frac{1-V_{ij}}{2\varkappa}\frac{\partial S_k}{\partial\varphi}+(j+1)\,B'q_{ij+1}+B'\frac{1-V_{ij}}{2\varkappa}(k+1)\,S_{k+1}+$$
$$+w_i(u_j\sin\theta+v_j\cos\theta)]+\sum_{i+j=n-2}(j+1)\,u_iw_{j+1}\sin\theta=0 \tag{2.4}$$

$$\sum_{i+j+k=n}\left\{R_0b_{ijk}+B\left[\frac{\varkappa-1}{2}(1-V_{ij})\frac{\partial w_k}{\partial\theta}-w_iw_j\frac{\partial w_k}{\partial\theta}-w_i\frac{\partial q_{jk}}{\partial\theta}\right]\right\}+$$

$$+\sum_{i+j+k=n-1}\Big\{b_{ijk}\sin\theta + R_0 c_{ijk} + \frac{\varkappa-1}{2}(1-V_{ij})\frac{\partial w_k}{\partial\varphi} - w_i w_j\frac{\partial w_k}{\partial\varphi} - \frac{w_i}{2}\frac{\partial q_{jk}}{\partial\varphi} -$$

$$- B'(k+1)\left[\frac{\varkappa-1}{2}(1-V_{ij})\,w_{k+1} - w_i w_j w_{k+1} - w_i q_{jk+1}\right] +$$

$$+\frac{\varkappa-1}{2}(1-V_{ij})(u_k\sin\theta + v_k\cos\theta)\Big\} + \sum_{i+j+k=n-2} c_{ijk}\sin\theta = 0 \qquad (2.5)$$

$$a_{ijk} = (k+1)\left(w_i w_{k+1} + v_i v_{k+1} + \frac{1-V_{ij}}{2\varkappa}S_{k+1}\right), \qquad q_{ij} = u_i u_j + v_i v_j, \quad V_{ij} = u_i u_j + v_i v_j + w_i w_j$$

$$b_{ijk} = \frac{1}{2}(\varkappa-1)(1-V_{ij})\left(\frac{\partial v_k}{\partial\theta} + u_k\right) - \frac{v_k}{2}\frac{\partial q_{ij}}{\partial\theta} - w_i v_j\frac{\partial w_k}{\partial\theta}$$

$$c_{ijk} = (k+1)\left[{}^1/_2(\varkappa-1)(1-V_{ij})\,u_{k+1} - u_i q_{jk+1} - w_i u_j w_{k+1}\right]$$

Equations (2.2) to (2.5) represent ordinary differential equations with respect to the independent variable θ with the parameter φ.

3. For the zero-order terms in series (2.1) we have, with $n = 0$ in (2.2) to (2.5)

$$\left(\frac{\partial u_0}{\partial\theta} - v_0\right)(Bw_0 + R_0 v_0) = 0, \qquad (Bw_0 + R_0 v_0)\frac{\partial S_0}{\partial\theta} = 0 \qquad (3.1)$$

$$R_0 v_0\frac{\partial w_0}{\partial\theta} - \left(u_0\frac{\partial u_0}{\partial\theta} + v_0\frac{\partial v_0}{\partial\theta}\right)B = \frac{1-V_0^2}{2\varkappa}\frac{\partial S_0}{\partial\theta}B \qquad (V_0^2 = u_0^2 + v_0^2 + w_0^2)$$

$$R_0 c_0^2\frac{\partial v_0}{\partial\theta} - R_0 v_0\left(u_0\frac{\partial u_0}{\partial\theta} + v_0\frac{\partial v_0}{\partial\theta}\right) + [B(c_0^2 - w_0^2) - R_0 w_0 v_0]\frac{\partial w_0}{\partial\theta} -$$

$$- Bw_0\left(u_0\frac{\partial u_0}{\partial\theta} + v_0\frac{\partial v_0}{\partial\theta}\right) + c_0^2 u_0 R_0 = 0 \qquad (c_0^2 = {}^1/_2(\varkappa-1)(1-V_0^2))$$

From the first two equations of system (3.1) we find

$$v_0 = \partial u_0/\partial\theta, \qquad S_0 = S_0(\varphi) \qquad (3.2)$$

Then the third equation (3.1) reduces to the form

$$\frac{\partial w_0}{\partial\theta} - \frac{B}{R_0}\left(u_0 + \frac{\partial v_0}{\partial\theta}\right) = 0 \qquad (3.3)$$

The fourth equation also simplifies and splits into two

$$u_0 + \frac{\partial v_0}{\partial\theta} = 0, \qquad \left(1 + \frac{B}{R_0^2}\right)c_0^2 - \left(v_0 + \frac{B}{R_0}w_0\right)^2 = 0 \qquad (3.4)$$

Equation (3.3) can be integrated

$$w_0 + u_0\frac{B'}{R_0} - v_0\frac{B}{R_0} = Z_0 w_0^+, \qquad Z_0 = \left[1 + \left(\frac{x_0^{\cdot}}{R_0}\right)^2 + \left(\frac{R_0^{\cdot}}{R_0}\right)^2\right]^{1/2} \qquad (3.5)$$

This equation shows that the projection of the velocity $w_0^+ = w_0^+(\varphi)$ in the direction of the tangent to the line L does not change under rotation of the stream about the curve L. Consequently for the zero-order terms in series (2.1) there are two different kinds of solution; we will distinguish the solution of type A if the zero-order terms of series (2.1) satisfy the first of equations (3.4), and the solution of type B if it satisfies the second.

Integrating equations (3.2) and (3.3) and the first of equations (3.4), we obtain for the zero-order terms of type A the following expressions:

$$w_0 = w_0(\varphi), \quad S_0 = S_0(\varphi), \quad u_0 = q_0 \cos(\theta - \delta), \quad v_0 = -q_0 \sin(\theta - \delta) \tag{3.6}$$

Here q_0 and δ will be functions of the angle φ. The solution (3.6) describes in each plane φ = const a flow with constant velocity V_0 directed at an angle $\delta(\varphi)$ to the x-axis. Subsequent terms of series (2.1) will take into account both the nonuniformity of the oncoming flow and the form of the stream surface on the line L.

Substituting relations (3.2) and (3.5) into the second equation (3.4), we obtain for the determination of the coefficient u_0 of the solution of type B the linear second-order equation

$$\begin{gathered}\gamma\left(1+\frac{B^2}{R_0^2}\right)^2\left(\frac{\partial u_0}{\partial\theta}\right)^2 + (Z_0^2 + \gamma B^2 B'^2)\, u_0^2 - \qquad \left(\gamma = \frac{\varkappa+1}{\varkappa-1}\right) \\ - 2\gamma B B'(1+B^2)\, u_0 \frac{\partial u_0}{\partial\theta} + 2\gamma B(1+B^2)\, w_0^+ Z_0 \frac{\partial u_0}{\partial\theta} - \\ - 2B'(1+\gamma B^2)\, w_0^+ Z_0 u_0 + w_0^{+2} Z_0^2 (1+\gamma B^2) - (1+B^2) = 0\end{gathered} \tag{3.7}$$

The solution of this equation is the function

$$u_0 = Z_0^{-1}\left(w_0^+ B' + \sqrt{1 - w_0^{+2}}\,\sqrt{1+B^2}\cos\beta\right) \tag{3.8}$$

$$\beta = 1/\sqrt{\gamma}\,\tan^{-1}\left[Z_0^{-1}\tan(\theta+\nu)\right] + a_0(\varphi), \qquad \tan\nu = x_0^{\cdot} / R_0^{\cdot}$$

Here $a_0(\varphi)$ is an arbitrary function. From equation (3.2) we find

$$v_0 = Z_0^{-1}\left[-w_0^+ B + \frac{BB'\sqrt{1-w_0^{+2}}}{\sqrt{1+B^2}}\cos\beta - 1/\sqrt{\gamma}\,\frac{\sqrt{1-w_0^{+2}}}{\sqrt{1+B^2}}\sin\beta\right] \tag{3.9}$$

Equation (3.7) and its solution are considerably simplified if in place of the plane φ = const we seek the solution in a plane perpendicular to the curve L at the given point. In order to transform to this new plane we carry out the following change of the variable θ and the unknown functions u_0, v_0 and w_0:

$$\theta^+ = \tan^{-1}\, [Z_0^{-1}\tan(\theta+\nu)], \qquad v_0 = Z_0^{-1}\left(\frac{Z_0}{\sqrt{1+B^2}}\, v_0^+ - Bw_0^+ + \frac{BB'}{\sqrt{1+B^2}}\, u_0^+\right) \tag{3.10}$$

$$u_0 = Z_0^{-1}(B'w_0^+ + \sqrt{1+B^2}\, u_0^+), \qquad w_0 = Z_0^{-1}\left(\frac{BZ_0}{\sqrt{1+B^2}}\, v_0^+ + w_0^+ - \frac{B'}{\sqrt{1+B^2}}\, u_0^+\right)$$

Here the plus superscript indicates components of velocity and functions in the new plane. Performing the transformation (3.10) in equations (3.2), (3.3) and (3.7), we obtain

$$\frac{\partial w_0^+}{\partial\theta^+} = 0, \qquad u_0^+ = \frac{\partial v_0^+}{\partial\theta^+}, \qquad \gamma\left(\frac{\partial u_0^+}{\partial\theta^+}\right)^2 + u_0^{+2} + w_0^{+2} - 1 = 0 \tag{3.11}$$

The solution of this system is the functions

$$u_0^+ = \sqrt{1-w_0^{+2}}\cos(1/\sqrt{\gamma}\,\theta^+ + a_0), \qquad w_0^+ = w_0^+(\varphi) \tag{3.12}$$
$$v_0^+ = -1/\sqrt{\gamma}\sqrt{1-w_0^{+2}}\sin(1/\sqrt{\gamma}\,\theta^+ + a_0)$$

$S_0 = S_0(\varphi)$ also remains valid, because the entropy does not depend upon the choice of coordinate system.

For $w_0^+ = 0$ relation (3.12) agrees with the well-known Prandtl-Meyer solution [1], representing a centered expansion wave in plane flow; consequently the solution of type B represents a three-dimensional expansion flow in the vicinity of the curve L.

If V_{01} is the magnitude of the velocity ahead of the turning of the stream, and the angle θ^+ is measured from the direction of the velocity up to the turning of the stream at the line L then, taking (3.12) into account, we have

$$V_{01}^2 = u_{01}^{+2} + v_{01}^{+2} + w_{01}^{+2} = 1 - (1 - w_0^{+2})\sin^2(1/\sqrt{\gamma}\,\theta_{01}^+ + a_0)$$
$$v_{01}^+/u_{01}^+ = \tan\theta_0^+ = -1/\sqrt{\gamma}\tan(1/\sqrt{\gamma}\,\theta_{01}^+ + a_0)$$

From these relations we find

$$a_0 = \sin^{-1}\frac{\sqrt{\gamma}\, c_{01}}{\sqrt{1-w_0^{+2}}} - \tan^{-1}\frac{c_{01}}{\sqrt{1-w_{01}^{+2} - {}^1/_2(\varkappa+1)(1-V_{01}^2)}}$$

where c_{01} is the speed of sound. In order that a real function a_0 exists, it is necessary that the inequality

$$1 - w_0^{+2} - {}^1/_2(\varkappa+1)(1-V_{01}^2) \geqslant 0$$

be satisfied, or

$$w_0^+ \leqslant \sqrt{1/2\,(\varkappa+1)\,V_{01}^2 - 1/2\,(\varkappa-1)} = V_{01}\cos\mu_1 \tag{3.13}$$

where μ_1 is the Mach angle ahead of the turning of the flow. Consequently condition (3.13) shows that a solution of type B exists only if the Mach cone issuing from a point of the curve contains no other point of the curve; that is, the line L must be a "supersonic edge".

4. Integration of the system of equations (2.2) to (2.5) successively for $n = 1, 2, 3, \ldots$ can be carried out by any numerical method. However in the case when L is the arc of a circumference the equations simplify considerably, and their solution can be found by quadrature.

If L is the arc of a circumference, $B = B' = 0$.

We extract from equations (2.2) to (2.5) terms with the unknown functions u_n, v_n, w_n and S_n, but we designate the remaining terms, which are known from the preceding solution, by Φ_n, G_n, H_n and F_n respectively. Calculation gives

$$R_0 v_0\left(\frac{\partial u_n}{\partial\theta} - v_n\right) - n\left(w_0 w_n R_0 + R_0 v_0 v_n + R_0\,\frac{1-V_0^2}{2\varkappa}\,S_n\right) + \Phi_n = 0 \tag{4.1}$$

$$v_0 R_0\,\frac{\partial S_n}{\partial\theta} + nR_0 u_0 S_n + G_n = 0 \tag{4.2}$$

$$v_0 R_0\,\frac{\partial w_n}{\partial\theta} + nR_0 u_0 w_n + H_n = 0 \tag{4.3}$$

$$\begin{aligned}
&R_0\Big[\frac{\varkappa-1}{2}\,(1-V_0^2)\left(\frac{\partial v_n}{\partial\theta} + u_n\right) - v_n\left(u_0\,\frac{\partial u_0}{\partial\theta} + v_0\,\frac{\partial v_0}{\partial\theta}\right) - \\
&\quad - w_0 v_0\,\frac{\partial w_n}{\partial\theta} - (\varkappa-1)\,(u_0 u_n + v_0 v_n + w_0 w_n)\left(\frac{\partial v_0}{\partial\theta} + u_0\right) - \\
&\quad - v_0\left(u_0\,\frac{\partial u_n}{\partial\theta} + v_0\,\frac{\partial v_n}{\partial\theta} + u_n\,\frac{\partial u_0}{\partial\theta} + v_n\,\frac{\partial v_0}{\partial\theta}\right)\Big] + \\
&+ nR_0\Big[\frac{\varkappa-1}{2}\,(1-V_0^2)\,u_n - u_0^2 u_n - v_0 u_0 v_n - w_0 u_0 w_n\Big] + F_n = 0
\end{aligned} \tag{4.4}$$

Equations (4.2) and (4.3) are integrated independently of the other equations. In the case of an expanding flow, using expressions (3.12) for u_0 and v_0, we obtain

$$\begin{aligned}
S_n &= \sin^{n\delta}\beta\left[\frac{\delta}{\sqrt{1-w_0^2}}\int_{\beta_0}^{\beta}\frac{G_n}{\sin^{n\delta+1}\beta}\,d\beta + A_n(\varphi)\right] \\
w_n &= \sin^{n\delta}\beta\left[\frac{\delta}{\sqrt{1-w_0^2}}\int_{\beta_0}^{\beta}\frac{H_n}{\sin^{n\delta+1}\beta}\,d\beta + B_n(\varphi)\right]
\end{aligned} \tag{4.5}$$

where A_n and B_n are arbitrary functions. Substituting equations (4.1) to (4.3) into equation (4.4) and using (3.12), we obtain

$$\sin 2\beta \frac{\partial u_n}{\partial \theta} - \left[\frac{(n+1)\varkappa - 1}{\sqrt{\varkappa^2 - 1}} \cos 2\beta + \frac{n - 1 + \varkappa}{\sqrt{\varkappa^2 - 1}}\right] u_n + \Omega_n = 0 \qquad (4.6)$$

Here

$$\Omega_n = \frac{\sqrt{\delta}}{1 - w_0^2}\left[u_0 v_0 w_n \left(n - \frac{2}{\delta}\right) + n u_0 \frac{1 - V_0^2}{2\varkappa} S_n - \frac{\Phi_n u_0 - w_0 H_n - F_n}{R_0}\right]$$

The solution of equation (4.6) is the function

$$u_n = \psi_n \tan \beta \left[-\frac{\sqrt{\delta}}{2}\int_{\beta_0}^{\beta} \frac{\Omega_n}{\psi_n \sin^2 \beta}\, d\beta + E_n(\varphi)\right] \qquad (\psi_n = \sin^{n\delta/2}\beta \cos^{1+n/2}\beta) \quad (4.7)$$

The value of v_n is found from equation (4.1). Equation (4.6) and its solution (4.7) agree for $w_n = S_n = 0$ with the solution of Shmyglevskii [1] for axisymmetric supersonic flow.

In case A equations (4.2) and (4.3) have the solution

$$\begin{aligned} S_n &= \sin^n(\theta - \delta)\left[\frac{1}{q_0 R_0}\int_{\theta_0}^{\theta} \frac{G_n}{\sin^{n+1}(\theta - \delta)}\, d\theta + L_n(\varphi)\right] \\ w_n &= \sin^n(\theta - \delta)\left[\frac{1}{q_0 R_0}\int_{\theta_0}^{\theta} \frac{H_n}{\sin^{n+1}(\theta - \delta)}\, d\theta + M_n(\varphi)\right] \end{aligned} \qquad (4.8)$$

From system (4.1) to (4.4) we find the linear equation for u_n

$$\begin{aligned} &\left(c_0^2 - \frac{1}{2} q_0^2 + \frac{1}{2} q_0^2 \cos 2\psi\right) R_0 \frac{\partial^2 u_n}{\partial \theta^2} + n R_0 q_0^2 \sin 2\psi \frac{\partial u_n}{\partial \theta} + \\ &+ (n+1) R_0 u_n \left[(n+1)\left(c_0^2 - \frac{1}{2} q_0^2\right) - \frac{n-1}{2} q_0^2 \cos 2\psi\right] + P_n = 0 \end{aligned} \qquad (4.9)$$

Here

$$\begin{aligned} P_n = n(n-1) R_0 u_0 \frac{c_0^2}{v_0^2}\left(v_0 w_n + \frac{1 - V_0^2}{2\varkappa} S_n\right) + \frac{c_0^2 - v_0^2}{v_0^2} n \left(w_0 H_n + \frac{1 - V_0^2}{2\varkappa} G_n\right) + \\ + \left(\frac{c_0^2}{v_0^2} - n\right) u_0 \Phi_n + \frac{c_0^2 - v_0^2}{v_0} \frac{\partial \Phi_n}{\partial \theta} + (n+1)(F_n + v_0 H_n) \end{aligned}$$

Equation (4.9) is satisfied by the functions

$$u_n = y_n\left[R_n + T^n \int_{\theta_0}^{\theta} \frac{T^n}{y_n^2}\, d\theta - \int_{\theta_0}^{\theta}\left(\frac{T^n}{y_n^2}\int_{\theta_0}^{\theta} \frac{P_n y_n}{T^{n+1}}\, d\theta\right) d\theta\right], \qquad y_n = \xi^{n+1}\left(1 + \sum_{m=1}^{\infty} g_m \xi^m\right) \qquad (4.10)$$

$$\xi = \frac{1 - M_0^2 \sin^2 \psi}{M_0^2 - 1}, \qquad T = c_0^2 - \frac{1}{2} q_0^2 + \frac{1}{2} q_0^2 \cos 2\psi, \quad g_1 = -\frac{n+1}{4}(M_0^2 - 2)$$

$$2g_2\frac{n+3}{(1-M_0^2)}-q_1\left(1+\frac{1}{1-M_0^2}\right)\left[\frac{3}{2}(n+2)-\frac{n}{4}(n+1)\right]+(n+1)+\frac{n^2-1}{4}=0$$

The remaining g_m are determined from the recurrence formulas

$$g_m\frac{m(m+n+1)}{1-M_0^2}-g_{m-1}\left(1+\frac{1}{1-M_0^2}\right)[(n+m)(m-{}^1/_2)-$$

$$-{}^1/_4\,n(n+1)]+g_{m-2}[(m+n-1)(m-1)+(n^2-1)/4]=0 \qquad (m=3,4,\ldots)$$

The functions v_n are then found from equation (4.1).

5. The coefficients of the series representing the solution of the system of equations (1.1) in form (2.1) are determined to within arbitrary functions of the variable φ, which must be found from the initial or boundary conditions. As an example we show in one such case a possible way of finding the unknown functions.

Let the flow ahead of the curve L be known, and turning of the flow by a certain angle take place at the curve. We consider the case of expanding flow. We represent the characteristic surface that passes through L and corresponds to the flow ahead of the turning of the stream by means of the series

$$\theta=\alpha_0(\varphi)+\alpha_1(\varphi)r+\alpha_2(\varphi)r^2+\ldots \tag{5.1}$$

We also write the parameters of the flow on this surface in the form of series; we choose as an example the component v_r

$$v_r=\beta_0(\varphi)+\beta_1(\varphi)r+\beta_2(\varphi)r^2+\ldots \tag{5.2}$$

On the other hand, v_r can be developed in a Taylor series

$$v_r=\sum_{j=0}^{\infty}\left(\frac{\partial^{(j)}v_r}{\partial\theta^j}\right)_{\theta=\alpha_0}\frac{(\theta-\alpha_0)^j}{j!} \tag{5.3}$$

From series (2.1) we find

$$\frac{\partial^{(j)}v_r}{\partial\theta^j}=\sum_{m=0}^{\infty}\frac{\partial^{(j)}u_m}{\partial\theta^j}r^m \tag{5.4}$$

and from (5.1)

$$(\theta-\alpha_0)^j=\left(\sum_{i=1}^{\infty}\alpha_i r^i\right)^j=\sum_{k=1}^{\infty}\left(\sum_{i_1+i_2+\ldots+i_j=k}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_j}\right)r^k \tag{5.5}$$

Substituting (5.4) and (5.5) into series (5.2), we obtain

$$v_r = \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{m=0}^{\infty} \sum_{k=j} \left(\frac{\partial^{(j)} u_m}{\partial\theta^j}\right)_{\theta=\alpha_0} \left(\sum_{i_1+i_2+\ldots+i_j=k} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_j}\right) r^{m+k} \qquad (5.6)$$

Equating coefficients of like powers of r in series (5.2) and (5.6), we obtain the equation

$$\beta_n = \sum_{j=0}^{n} \frac{1}{j!} \sum_{m+k=n} \left(\frac{\partial^{(j)} u_m}{\partial\theta^j}\right)_{\theta=\alpha_0} \left(\sum_{i_1+i_2+\ldots+i_j=k} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_j}\right) \qquad (5.7)$$

where the functions β_n and α_i are known, but in the expressions for the derivatives $\partial^{(j)} u_m / \partial\theta^j$ there appear the unknown functions $E_1, \ldots, E_{n-1}$, E_n and derivatives with respect to θ of $E_n, \ldots, E_{n-1}$, so that from equation (5.7) it is possible to find successively all the functions E_n. The other derivatives of the functions A_n and B_n are also found in the same way.

With the proposed method, using the solution of type A, it is also possible to seek a solution in the vicinity of a curve L when it is the line of origin of a shock wave. The method may be used both for analytic investigation and for numerical methods of solution of three-dimensional supersonic flow of gas to find the solution in the vicinity of singular points.

With regard to the convergence of series (2.1) it should be mentioned that in the case of axisymmetric potential flow and analytic initial conditions the convergence of these series was shown by Dorodnitsyn [2]. In the general case this question remains open.

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Translated by M.D.v.D.